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SOME IDENTITIES OF DERANGEMENT NUMBERS

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Abstract. In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position. That is, a derangement is a permutation that has no fixed points. The number of derangements of an n-element set is called the n-th derangement number. In this paper, we study some identities involving derangement numbers, r-derangement numbers and some other special numbers which are derived from various generating functions.

1. Introduction

A derangement is a permutation with no fixed points. In other words, a derangement of a set leaves no elements in the original place. The number of derangements of a set of size n, denoted d_n, is called the n-th derangement number (see [1,2,10]).

The first few derangement numbers, starting from $n = 0$, are 1,0,2,9,44,265, 1854,14833,133496,4684570, $\cdots$. For $n \geq 0$, the derangement numbers are given by

$$\begin{aligned} d_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n\binom{n}{n}0! \\ &= \sum_{k=0}^{n}\binom{n}{k}(n-k)!(-1)^k = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}, \quad (\text{see } [1,2,10]). \end{aligned} \tag{1.1}$$

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From (1.1), we can derive the generating function of derangement numbers which are given by

$$\begin{aligned}\sum_{n=0}^{\infty} d_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{n=k}^{\infty} t^n = \frac{1}{1-t} e^{-t}.\end{aligned} \tag{1.2}$$

By (1.2), we easily get the following recurrence relations:

$$d_n = n \cdot d_{n-1} + (-1)^n, \ (n \geq 1), \tag{1.3}$$

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \ (n \geq 2), \ (see[1, 2, 10]), \tag{1.4}$$

and

$$\sum_{k=0}^{n} d_k \binom{n}{k} = n!, \ (n \geq 0). \tag{1.5}$$

By (1.1), we easily get

$$d_n - n! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} - n! = n! \sum_{k=1}^{n} \frac{(-1)^k}{k!}, \ (n \geq 0), \tag{1.6}$$

and

$$\lim_{n \to \infty} \frac{d_n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}. \tag{1.7}$$

The number of arrangements of any subset of n distinct objects is the number of one-to-one sequences that can be formed from any subset of n distinct objects.

The number of arrangements of any subject of n distinct objects are called arrangement numbers which are denoted by a_n, $(n \geq 0)$. The arrangement numbers are given by

$$\begin{aligned}a_n &= n! + \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + \binom{n}{n} 0! \\ &= \sum_{k=0}^{n} \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{1}{k!}, \ (n \geq 0).\end{aligned} \tag{1.8}$$

From (1.8), we note that

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{1}{k!} \right) \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} t^k$$
$$= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \left(\sum_{n=0}^{\infty} t^n \right) = \frac{1}{1-t} e^t. \tag{1.9}$$

By (1.9), we easily get

$$a_0 = 1, a_n = n a_{n-1} + 1, \ (n \geq 1), \tag{1.10}$$

and

$$a_n = (n-1)(a_{n-1} + a_{n-2}) + 2, \ (n \geq 2). \tag{1.11}$$

For $0 \leq r \leq n$, the r-derangement numbers, denoted $d_n^{(r)}$, are the number of derangements on $n+r$ elements under the restriction that the first r elements are in disjoint cycles. It is known that the generating function of the r-derangement numbers is given by

$$\sum_{n=0}^{\infty} d_n^{(r)} \frac{t^n}{n!} = \frac{t^r}{(1-t)^{r+1}} e^{-t}, \quad \text{(see [10])}. \tag{1.12}$$

As is well known, the Bell numbers are defined by

$$e^{e^t - 1} = \sum_{n=0}^{\infty} Bel_n \frac{t^n}{n!}, \quad \text{(see [5, 9])}. \tag{1.13}$$

On the other hand,

$$e^{e^t-1} = \frac{1}{e} e^{e^t} = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} e^{kt}$$
$$= \frac{1}{e} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k^n}{k!} \frac{t^n}{n!}. \tag{1.14}$$

Thus, by (1.13) and (1.14), we get

$$Bel_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}, \ (n \geq 0), \quad \text{(see [3, 4, 5])}. \tag{1.15}$$

For $n \geq 0$, the Stirling numbers of the first kind are defined as

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 1),\ (x)_0 = 1. \tag{1.16}$$

The Stirling numbers of the second kind are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l,\ (n \geq 0), \quad \text{(see [9])}. \tag{1.17}$$

The ordered Bell numbers are given by the generating function

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}, \quad \text{(see [3])}. \tag{1.18}$$

It is well known that the Euler numbers are defined by

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{(see [3 − 9])}. \tag{1.19}$$

From (1.17) and (1.18), we have

$$b_n = \sum_{m=0}^{n} m! S_2(n,m),\ (n \geq 0). \tag{1.20}$$

In this paper, we investigate some properties of several special numbers. Then we proceed to give some new identities involving derangement numbers, r-derangement numbers and some other special numbers like Bell numbers, Euler numbers and Stirling numbers of the first and second kind which are derived from various generating functions.

2. Some identities of derangement numbers.

From (1.13), we note that

$$\begin{aligned} e^{-t} &= \sum_{k=0}^{\infty} Bel_k \frac{1}{k!} \log^k(1-t) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} Bel_k S_1(n,k)(-1)^n \right) \frac{t^n}{n!} \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} e^{-t} &= \left(\frac{1}{1-t}e^{-t}\right)(1-t) = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!}(1-t) \\ &= d_0 + \sum_{n=1}^{\infty}(d_n - nd_{n-1}\frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

Thus, by (2.1) and (2.2), we get

$$\begin{aligned} d_0 = Bel_0 = 1, &\sum_{k=0}^{n} Bel_k S_1(n,k) = (-1)^n(d_n - nd_{n-1}), \ (n \geq 1). \\ \text{Indeed,} \qquad &\sum_{k=0}^{n} Bel_k S_1(n,k) = 1, \ (n \geq 1). \end{aligned} \tag{2.3}$$

By replacing t by $-e^t + 1$in (1.2), we get

$$\begin{aligned} e^{-t}e^{(e^t-1)} &= \sum_{k=0}^{\infty} d_k(-1)^k \frac{1}{k!}(e^t-1)^k \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k d_k S_2(n,k)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned} e^{-t}e^{(e^t-1)} &= \left(\sum_{l=0}^{\infty}(-1)^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} Bel_m \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} Bel_m (-1)^{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5), we obtain the following lemma.

Lemma 2.1. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}(-1)^k d_k S_2(n,k) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} Bel_k.$$

Let us replace t by $-e^t$ in (1.2). Then we have

$$\begin{aligned}\frac{1}{e^t+1}e^{e^t} &= \sum_{k=0}^{\infty}\frac{d_k}{k!}(-1)^k e^{tk}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{d^k}{k!}(-1)^k k^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned}\frac{1}{e^t+1}e^{e^t} &= \frac{e}{2}\left(\frac{2}{e^t+1}\right)e^{e^t-1}\\ &= \frac{e}{2}\left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}Bel_m\frac{t^m}{m!}\right)\\ &= \frac{e}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}Bel_k E_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}Bel_k E_{n-k} = \frac{2}{e}\sum_{k=0}^{\infty}\frac{d^k}{k!}(-1)^k k^n.$$

Remark. We note that

$$\begin{aligned}\frac{1}{e^t+1}e^{e^t} &= \frac{e}{2}\left(\frac{2}{e^t+1}\right)e^{e^t-1} = \frac{e}{2}\left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}(e^t-1)^m\right)\\ &= \frac{e}{2}\left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}S_2(k,m)\frac{t^k}{k!}\right)\\ &= \frac{e}{2}\left(\sum_{l=0}^{\infty}E_l\frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}S_2(k,m)\right)\frac{t^k}{k!}\right)\\ &= \frac{e}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}S_2(k,m)E_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By (2.6) and (2.8), we get

$$\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}S_2(k,m)E_{n-k} = \frac{2}{e}\sum_{k=0}^{\infty}(-1)^k\frac{d^k}{k!}k^n \tag{2.9}$$

From (1.2) and (1.9), we have

$$\begin{aligned}\frac{1}{1-x^2} &= \left(\frac{1}{1-x}e^{-x}\right)\left(\frac{1}{1+x}e^{x}\right) = \left(\sum_{l=0}^{\infty} d_l \frac{x^l}{l!}\right)\left(\sum_{m=0}^{\infty} d_m(-1)^m \frac{x^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} d_l d_{n-l}(-1)^{n-l}\right)\frac{x^n}{n!}.\end{aligned} \tag{2.10}$$

On the other hand,

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2m} = \sum_{n=0}^{\infty}(2n)!\frac{x^{2n}}{(2n)!}. \tag{2.11}$$

By (2.10) and (2.11), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(2n)!\frac{x^{2n}}{(2n)!} &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} d_l d_{n-l}(-1)^{n-l}\right)\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{2n}\binom{2n}{l} d_l d_{2n-l}(-1)^{l}\right)\frac{x^{2n}}{(2n)!} \\ &\quad + \sum_{n=0}^{\infty}\left(\sum_{l=0}^{2n+1}\binom{2n+1}{l} d_l d_{2n+1-l}(-1)^{l-1}\right)\frac{x^{2n+1}}{(2n+1)!}.\end{aligned} \tag{2.12}$$

Comparing the coefficients on both sides of (2.12), we have

$$\sum_{l=0}^{2n+1}\binom{2n+1}{l} d_l d_{2n+1-l}(-1)^{l-1} = 0, \tag{2.13}$$

and

$$\sum_{l=0}^{2n}\binom{2n}{l} d_l d_{2n-l}(-1)^{l} = (2n)!. \tag{2.14}$$

By (2.14), we easily get

$$\sum_{l=0}^{2n}\left(\frac{d_l}{l!}\right)\cdot\left(\frac{d_{2n-l}}{(2n-l)!}\right)(-1)^l = 1. \tag{2.15}$$

Therefore, by (2.13) and (2.15), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\sum_{l=0}^{2n}\left(\frac{d_l}{l!}\right)\cdot\left(\frac{d_{2n-l}}{(2n-l)!}\right)(-1)^l = 1,$$

and

$$\sum_{l=0}^{2n+1}\left(\frac{d_l}{l!}\right)\cdot\left(\frac{d_{2n+1-l}}{(2n+1-l)!}\right)(-1)^l=0.$$

For $r\in\mathbb{N}$, we observe that

$$\begin{aligned}\left(\frac{1}{1-t}\right)^r&=\left(\frac{1}{1-t}\right)^r e^{-rt}e^{rt}=\underbrace{\left(\frac{1}{1-t}e^{-t}\right)\times\left(\frac{1}{1-t}e^{-t}\right)\times\left(\frac{1}{1-t}e^{-t}\right)}_{r-\text{times}}\times e^{rt}\\&=\left(\sum_{k=0}^{\infty}\left(\sum_{l_1+\cdots+l_r=k}\binom{k}{l_1,\cdots,l_r}d_{l_1}d_{l_2}\cdots d_{l_r}\right)\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}r^m\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{l_1+\cdots+l_r=k}\binom{k}{l_1,\cdots,l_r}\binom{n}{k}r^{n-k}d_{l_1}d_{l_2}\cdots d_{l_r}\right\}\frac{t^n}{n!},\end{aligned}\tag{2.16}$$

where $\binom{n}{l_1,\cdots,l_r}=\frac{n!}{l_1!l_2!\cdots l_r!}$. On the other hand,

$$\left(\frac{1}{1-t}\right)^r=\sum_{n=0}^{\infty}\binom{n+r-1}{n}t^n=\sum_{n=0}^{\infty}n!\binom{n+r-1}{n}\frac{t^n}{n!}.\tag{2.17}$$

From (2.16) and (2.17), we have

$$\binom{n+r-1}{n}=\frac{1}{n!}\sum_{k=0}^{n}\sum_{l_1+\cdots+l_r=k}\binom{k}{l_1,\cdots,l_r}\binom{n}{k}r^{n-k}d_{l_1}d_{l_2}\cdots d_{l_r}.\tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, $r\in\mathbb{N}$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{r-1}{n-k}=\frac{1}{n!}\sum_{k=0}^{n}\sum_{l_1+\cdots+l_r=k}\binom{k}{l_1,\cdots,l_r}\binom{n}{k}r^{n-k}d_{l_1}d_{l_2}\cdots d_{l_r}.$$

Replacing t by e^t-1 in (1.2), we get

$$\begin{aligned}\frac{1}{2-e^t}e^{-(e^t-1)}&=\sum_{k=0}^{\infty}d_k\frac{1}{k!}(e^t-1)^k\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}d_kS_2(n,k)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.19}$$

On the other hand,

$$\begin{aligned}\frac{1}{2-e^t}e^{-(e^t-1)} &= \left(\sum_{l=0}^{\infty} b_l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!}(e^t-1)^m\right)\\ &= \left(\sum_{l=0}^{\infty} b_l \frac{t^l}{l!}\right)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}(-1)^m S_2(k,m)\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(-1)^m S_2(k,m) b_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} d_k S_2(n,k) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(-1)^m S_2(k,m) b_{n-k}.$$

Now, we observe that

$$\left(\frac{1}{1-t}\right)^{2r} = \underbrace{\left(\frac{1}{1-t}e^{-t}\right)\times\left(\frac{1}{1-t}e^{t}\right)\times\cdots\times\left(\frac{1}{1-t}e^{-t}\right)\times\left(\frac{1}{1-t}e^{t}\right)}_{2r-\text{times}} \tag{2.21}$$

where r is natural number. By (2.21), we easily get

$$\begin{aligned}&\left(\frac{1}{1-t}\right)^{2r}\\ &= \left(\sum_{l_1=0}^{\infty}\frac{d_{l_1}}{l_1!}t^{l_1}\right)\left(\sum_{l_2=0}^{\infty}\frac{a_{l_2}}{l_2!}t^{l_2}\right)\cdots\left(\sum_{l_{2r-1}=0}^{\infty}\frac{d_{l_{2r-1}}}{l_{2r-1}!}t^{l_{2r-1}}\right)\left(\sum_{l_{2r}=0}^{\infty}\frac{a_{l_{2r}}}{l_{2r}!}t^{l_{2r}}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l_1+\cdots+l_{2r}=n}\binom{n}{l_1,\ldots,l_{2r}} d_{l_1}a_{l_2}\cdots d_{2r-1}a_{2r}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.22}$$

On the other hand,

$$\left(\frac{1}{1-t}\right)^{2r} = \sum_{n=0}^{\infty}\binom{n+2r-1}{n}x^n = \sum_{n=0}^{\infty}n!\binom{n+2r-1}{n}\frac{x^n}{n!}. \tag{2.23}$$

From (2.21), (2.22) and (2.23), we have

$$n!\binom{n+2r-1}{n} = \sum_{l_1+\cdots+l_{2r}=n} \binom{n}{l_1,\ldots,l_{2r}} d_{l_1}a_{l_2}\cdots d_{2r-1}a_{2r},$$

where $n \geq 0$, and $r \in \mathbb{N}$.
Therefore, we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, $r \in \mathbb{N}$. we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2r-1}{n-k} = \sum_{l_1+\cdots+l_{2r}=n}\left(\frac{d_{l_1}}{l_1!}\right)\left(\frac{a_{l_2}}{l_2!}\right)\cdots\left(\frac{d_{l_{2r-1}}}{l_{2r-1}!}\right)\left(\frac{a_{l_{2r}}}{l_{2r}!}\right).$$

From (1.12), we note that

$$\begin{aligned}\frac{t^r}{(1-t)^{r+1}}e^{-t} &= \left(t^r\sum_{l=0}^{\infty}\binom{-r-1}{l}(-t)^l\right)\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}t^m\right)\\ &= \left(\sum_{l=0}^{\infty}\binom{l+r}{l}t^{l+r}\right)\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}t^m\right)\\ &= \left(\sum_{l=r}^{\infty}\binom{l}{r}t^{l}\right)\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}t^m\right)\\ &= \sum_{n=r}^{\infty}\left(n!\sum_{l=r}^{n}\binom{l}{r}\frac{(-1)^{n-l}}{(n-l)!}\right)\frac{t^n}{n!},\end{aligned} \tag{2.24}$$

and

$$\frac{t^r}{(1-t)^{r+1}}e^{-t} = \sum_{n=0}^{\infty} d_n^{(r)}\frac{t^n}{n!}. \tag{2.25}$$

Comparing the coefficients on both sides of (2.24) and (2.25), we have

$$d_n^{(r)} = n!\sum_{l=r}^{n}\binom{l}{r}\frac{(-1)^{n-l}}{(n-l)!},\ (n \geq r), \tag{2.26}$$

and

$$d_0^{(r)} = d_1^{(r)} = \cdots = d_{r-1}^{(r)} = 0.$$

Therefore, by (2.26), we obtain the following proposition.

Proposition 2.7. *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$\frac{d_n^{(r)}}{n!} = \sum_{l=r}^{n}\binom{l}{r}\frac{(-1)^{n-l}}{(n-l)!},\ (n \geq r),$$

and

$$d_0^{(r)} = d_1^{(r)} = \cdots = d_{r-1}^{(r)} = 0.$$

By (1.2) and (1.12), we get

$$\begin{aligned}\sum_{n=0}^{\infty} d_n^{(r)} \frac{t^n}{n!} &= \left(\frac{t}{1-t}\right)^r \left(\frac{1}{1-t}e^{-t}\right) \\ &= \left(\sum_{l=r}^{\infty} \binom{l-1}{l-r} t^l\right) \left(\sum_{m=0}^{\infty} d_m \frac{t^m}{m!}\right) \\ &= \sum_{n=r}^{\infty} \left(n! \sum_{l=r}^{n} \binom{l-1}{l-r} \frac{d_{n-l}}{(n-l)!}\right) \frac{t^n}{n!}.\end{aligned} \tag{2.27}$$

Thus, by (2.27), w get

$$d_n^{(r)} = n! \sum_{l=r}^{n} \binom{l-1}{l-r} \frac{d_{n-l}}{(n-l)!}, \tag{2.28}$$

where $n \geq r$. Therefore, by (2.28), we obtain the following theorem.

Corollary 2.8. *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$\frac{d_n^{(r)}}{n!} = \sum_{l=r}^{n} \binom{l-1}{l-r} \frac{d_{n-l}}{(n-l)!},$$

and

$$d_0^{(r)} = d_1^{(r)} = \cdots = d_{r-1}^{(r)} = 0.$$

Now, we observe that

$$\begin{aligned} e^{-t} &= \frac{(1-t)^{r+1}}{t^r} \left(\sum_{k=r}^{\infty} d_k^{(r)} \frac{t^k}{k!}\right) \\ &= (1-t)^{r+1} \left(\sum_{k=0}^{\infty} d_{k+r}^{(r)} \frac{t^k}{(k+r)!}\right) \\ &= \left(\sum_{l=0}^{\infty} \binom{r+1}{l} (-1)^l t^l\right) \left(\sum_{k=0}^{\infty} \frac{d_{k+r}^{(r)}}{(k+r)!} t^k\right) \\ &= \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{d_{k+r}^{(r)}}{(k+r)!} (-1)^{n-k} \binom{r+1}{n-k}\right) \frac{t^n}{n!}. \end{aligned} \tag{2.29}$$

On the other hand,

$$e^{-t}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^n. \tag{2.30}$$

Therefore, by (2.29) and (2.30), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$, we have*

$$\frac{1}{n!}=\sum_{k=0}^{n}\frac{d_{k+r}^{(r)}}{(k+r)!}(-1)^k\binom{r+1}{n-k}.$$

From Proposition 2.7 and (1.12), we have

$$\begin{aligned}\frac{t^r}{(1-t)^{r+1}}e^{-t}&=\sum_{n=0}^{\infty}d_n^{(r)}\frac{t^n}{n!}=\sum_{n=r}^{\infty}d_n^{(r)}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}d_{n+r}^{(r)}\frac{n!}{(n+r)!}\frac{t^{n+r}}{n!}=\sum_{n=0}^{\infty}\frac{d_{n+r}^{(r)}}{\binom{n+r}{n}r!}\frac{t^{n+r}}{n!}.\end{aligned} \tag{2.31}$$

Thus, by (2.31), we get

$$\frac{1}{(1-t)^{r+1}}e^{-t}=\sum_{n=0}^{\infty}\frac{d_{n+r}^{(r)}}{\binom{n+r}{n}r!}\frac{t^n}{n!}. \tag{2.32}$$

On the other hand,

$$\begin{aligned}\frac{1}{(1-t)^{r+1}}e^{-t}&=\left(\frac{1}{1-t}\right)^r\left(\frac{1}{1-t}e^{-t}\right)\\&=\left(\sum_{l=0}^{\infty}\binom{r+l-1}{l}t^l\right)\left(\sum_{m=0}^{\infty}d_m\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(n!\sum_{l=0}^{n}\binom{r+l-1}{l}\frac{d_{n-l}}{(n-l)!}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.33}$$

From (2.32) and (2.33), we have

$$\frac{d_{n+r}^{(r)}}{r!\binom{n+r}{n}}=n!\sum_{l=0}^{n}\binom{r+l-1}{l}\frac{d_{n-l}}{(n-l)!},\ (n\geq 0). \tag{2.34}$$

By (2.35), we get

$$\begin{aligned} d_{n+r}^{(r)} &= r!\binom{n+r}{n} n! \sum_{l=0}^{n} \binom{r+l-1}{l} \frac{d_{n-l}}{(n-l)!} \\ &= (n+r)! \sum_{l=0}^{n} \binom{r+l-1}{l} \frac{d_{n-l}}{(n-l)!}. \end{aligned} \tag{2.35}$$

3. Further remark

For $r \in \mathbb{N}$, we have

$$\begin{aligned} \left(\frac{d}{dt}\right)^r \left(\frac{1}{1-t}\right) &= r! \frac{1}{(1-t)^{r+1}} = \frac{r!}{t^r} \frac{t^r e^{-t}}{(1-t)^{r+1}} e^t \\ &= r! \frac{e^t}{t^r} \sum_{n=r}^{\infty} d_n^{(r)} \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} d_{n+r}^{(r)} \frac{r! n!}{(n+r)!} \frac{t^n}{n!} \\ &= \left(\sum_{k=0}^{\infty} \frac{d_{k+r}^{(r)}}{\binom{k+r}{k}} \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} \frac{1}{m!} t^m \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{k+r}{k}} d_{k+r}^{(r)} \right) \frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} \left(\frac{d}{dt}\right)^r \left(\frac{1}{1-t}\right) &= \left(\frac{d}{dt}\right)^r \left(\frac{1}{1-t} e^{-t} e^t\right) \\ &= \left(\frac{d}{dt}\right)^r \left\{ \left(\sum_{m=0}^{\infty} d_m \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{1}{l!} t^l \right) \right\} \\ &= \left(\frac{d}{dt}\right)^r \left\{ \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} d_m \binom{n}{m} \right) \frac{t^n}{n!} \right\} \\ &= \sum_{n=r}^{\infty} \left(\sum_{m=0}^{\infty} d_m \binom{n}{m} \right) \frac{1}{(n-r)!} t^{n-r} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n+r} d_m \binom{n+r}{m} \right) \frac{t^n}{n!}. \end{aligned} \tag{3.2}$$

Therefore, by (3.1) and (3.2), we obtain the following theorem.

Theorem 3.1. *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{k+r}{k}} d_{k+r}^{(r)} = \sum_{m=0}^{n+r} d_m \binom{n+r}{m}.$$

Let

$$F = F(t) = \frac{1}{1-t} e^{-t}. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned} F^{(1)} = \frac{d}{dt} F(t) &= \frac{-e^{-t}(1-t) + e^{-t}}{(1-t)^2} = \frac{te^{-t}}{(1-t)^2} \\ &= \left(\frac{t}{1-t}\right) \frac{1}{1-t} e^{-t} = \frac{t}{1-t} F. \end{aligned} \tag{3.4}$$

Thus, by (3.4), we get

$$(1-t)F^{(1)} = tF. \tag{3.5}$$

From (3.5), we have

$$\begin{aligned} -F^{(1)} + (1-t)F^{(2)} &= F + tF^{(1)} = F + (t-1)F^{(1)} + F^{(1)} \\ &= F - (1-t)F^{(1)} + F^{(1)}, \end{aligned} \tag{3.6}$$

where $F^{(n)} = \left(\frac{d}{dt}\right)^n F(t)$, $(n \in \mathbb{N})$.
By (3.6), we get

$$-2F^{(1)} + (1-t)F^{(2)} = F - (1-t)F^{(1)} = F - tF = (1-t)F. \tag{3.7}$$

Thus, we note that

$$-2F^{(1)} + (1-t)F^{(2)} = (1-t)F. \tag{3.8}$$

Now, we take the derivative on both sides of (3.8).

$$-2F^{(2)} - F^{(2)} + (1-t)F^{(3)} = -F + (1-t)F^{(1)} = -(1-t)F. \tag{3.9}$$

Thus, by (3.9), we get

$$-3F^{(2)} + (1-t)F^{(3)} = -(1-t)F. \tag{3.10}$$

It is not difficult to show that

$$-4F^{(3)} + (1-t)F^{(4)} = (1-t)F, \tag{3.11}$$

and

$$-5F^{(4)} + (1-t)F^{(5)} = -(1-t)F. \tag{3.12}$$

Continuing this process, we have

$$(-1)^{N-1}(1-t)F = -(N+1)F^{(N)} + (1-t)F^{(N+1)}, \ (n \in \mathbb{N}), \tag{3.13}$$

where $F^{(N)} = \left(\frac{d}{dt}\right)^N F(t)$. From (3.3), we note that

$$(-1)^{N-1}(1-t)F = (-1)^{N-1}e^{-t} = \sum_{n=0}^{\infty}(-1)^{N-1-n}\frac{t^n}{n!}, \tag{3.14}$$

$$\begin{aligned} -(N+1)F^{(N)} &= -(N+1)\left(\frac{d}{dt}\right)^N \sum_{n=0}^{\infty} d_n \frac{t^n}{n!} \\ &= -(N+1)\sum_{n=N}^{\infty} d_n \frac{t^{n-N}}{(n-N)!} \\ &= -(N+1)\sum_{n=0}^{\infty} d_{n+N}\frac{t^n}{n!}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} (1-t)F^{(N+1)} &= (1-t)\left(\frac{d}{dt}\right)^{N+1}\sum_{n=0}^{\infty} d_n \frac{t^n}{n!} \\ &= (1-t)\sum_{n=N+1}^{\infty} d_n \frac{1}{(n-N-1)!}t^{n-N-1} \\ &= (1-t)\sum_{n=0}^{\infty} d_{n+N+1}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} d_{n+N+1}\frac{t^n}{n!} - \sum_{n=1}^{\infty} n d_{n+N}\frac{t^n}{n!} \\ &= d_{N+1} + \sum_{n=1}^{\infty}\left(d_{n+N+1} - n d_{n+N}\right)\frac{t^n}{n!}. \end{aligned} \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.17), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^{N-1-n}\frac{t^n}{n!}\\&=\Big(-(N+1)d_N+d_{N+1}\Big)+\sum_{n=1}^{\infty}\left(d_{n+N+1}-nd_{n+N}-(N+1)d_{n+N}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.17}$$

By comparing the coefficients on both sides of (3.17), we obtain the following theorem.

Theorem 3.2. *For $N \in \mathbb{N}$, we have*

$$(1) \quad d_{N+1}=(N+1)d_N+(-1)^{N-1}$$

and

$$(2) \quad d_{n+N+1}=n\cdot d_{n+N}+(N+1)d_{n+N}+(-1)^{N-1-n}.$$

References

1. L. Carlitz, *The number of derangements of a sequence with given specification,* Fibonacci Quart. **16** (1978), no. 3, 255-258.
2. R. J. Clarke, M. Sved, *Derangements and Bell numbers,* Math. Mag. **66** (1993), no. 5, 299-303.
3. G.-W. Jang, T. Kim, *Some identities of ordered Bell numbers arising from differential equation,* Adv. Stud. Contemp. Math (Kyungshang), **27** (2017), no. 3, 385-397.
4. L.-C. Jang, T. Kim, *Some identities of Bell polynomials,* J. Comput. Anal. Appl., **20** (2016), no. 3, 584-589.
5. D. S. Kim, T. Kim, *Some identities of Bell polynomials,* Sci. China Math., **58** (2015), no. 10, 2095-2104.
6. T. Kim, *Degenerate ordered Bell numbers and polynomials,* Proc. Jangjeon Math. Soc., **20** (2017), no. 2, 137-144.
7. T. Kim, *λ-analogue of Stirling numbers of the first kind,* Adv. Stud. Contemp. Math (Kyungshang), **27** (2017), no. 3, 423-429.
8. T. Kim, D. S. Kim, *On λ-Bell polynomials associated with umbral calculus,* Russ. J. Math. Phys., **24** (2017), no. 1, 69-78.
9. S. Roman, *The umbral calculus,* Pure and Applied Mathematics, 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, **1984**, x+193 pp.
10. C. Wang, P. Miska, I. Mezo, *The r-derangement numbers,* Discrete Math., **340**, (2017), no. 7, 1681-1692.

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